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Iterative-Order-Reduction Substructuring Method for Dynamic Condensation of Finite Element Models

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An iterative condensation method for both undamped and damped structures is presented. The iterative order-reduction method, which uses iterative forms of mass and transformation matrices to reduce the finite element system matrices, is a computationally efficient technique to obtain the lowest eigenvalues of undamped and classically damped finite element models. However, the iterative order-reduction method is not suitable for application in nonclassically damped systems, due to the absence of damping in the dynamic transformation matrix. In this paper, an improved iterative order-reduction method based on the state-space model is proposed for nonclassically damped systems by fully taking into account the effect of damping on the transformation matrix. By combining the original and improved iterative order-reduction schemes with the substructuring scheme, a new iterative order-reduction substructuring method is also proposed to overcome the problem of requiring more computational resources to construct the transformation matrix of large-scale finite element models. The proposed method may be applied to the reduction problems of large-scale undamped and damped finite element models with limited computer storage and high computational efficiency. Two numerical examples are provided to demonstrate the effectiveness of the proposed method.

Nomenclature

C = system damping matrix of order $n \times n$ in the displacement space
D = direction matrix of excitation force f in the displacement space
$\bar{D}$ = direction matrix of excitation force f in the state space
D_R = reduced direction matrix of excitation f in the displacement space
$\tilde{D}_R$ = reduced direction matrix of excitation f in the state space
f = excitation force
G = system matrix of order $2n \times 2n$ in the state space
G_G = reduced system matrix of order $2m \times 2m$ in the state space by the Guyan method
H = system matrix of order $2n \times 2n$ in the state space
H_d = reduced system matrix of order $2m \times 2m$ in the state space
H_G = reduced system matrix of order $2m \times 2m$ in the state space by the Guyan method
K = stiffness matrix of order $n \times n$ in the displacement space
K_G = reduced stiffness matrix by using the Guyan method in the state space
K_R = reduced stiffness matrix of order $m \times m$ in the displacement space
M = mass matrix of order $n \times n$ in the displacement space
M_d = reduced mass matrix in the displacement space
M_R = reduced mass matrix of order $m \times m$ in the displacement space
t = transformation matrix in the displacement and state space
T = combined transformation matrix in the displacement and state space
U = complex conjugate eigenvectors of order $2n \times 2m$ corresponding to the nonclassically damped system
$x, \dot{x}, \ddot{x}$ = displacement, velocity and acceleration vectors of order $n \times 1$ in the displacement space
$z, \dot{z}, \ddot{z}$ = displacement, velocity and acceleration vectors of order $2n \times 1$ in the state space
Λ = diagonal matrix containing the eigenvalues corresponding to the undamped system on the diagonal
Φ = eigenvectors of order $n \times m$ corresponding to the undamped system
Ω = eigenvalue matrix corresponding to the nonclassically damped system

Subscripts

m = retained coordinates of master modes
s = removed coordinates of slave modes

Superscripts

T = transposition of the matrix
j = jth substructure
k = kth iteration time
$*$ = complex conjugation

I. Introduction

MODERN finite element (FE) models have been widely developed to obtain good approximations of physical systems. They require much more computational resources to deal with the generalized eigenvalue problems when the number of degrees of freedom (DOF) becomes large. To reduce the computational cost and to use these models effectively in further dynamic analysis and optimization design, it is necessary to reduce the dimensions of system matrices while still retaining relevant characteristics. Many researchers have been interested in dealing with the eigenvalue problems of FE models while limiting computer storage and improving speed [1–3].

Dynamic condensation, an efficient model-reduction technique, can resolve these problems and obtain much smaller system matrices. This reduction strategy removes some DOF (slave DOF) from the

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original FE models and retains a much smaller set of DOF (master DOF). Since the static condensation techniques were proposed by Guyan [4] and Irons [5] in 1965, many kinds of reduction algorithms have been investigated to increase the accuracy of reduced FE models. For undamped systems, Paz [6] studied Guyan's method with shifted eigenvalues. O'Callaghan [7] developed an improved reduced system (IRS) method that provided a perturbation for the transformation from the static case by including the inertia terms as pseudostatic forces. Friswell et al. [8] extended the IRS method by obtaining the equivalent transformation with an iterative algorithm. A proof of convergence of this method was given by Friswell et al. [9]. Xia and Lin [10] proposed an efficient technique that retains all the inertia terms associated with the removed DOF. For nonclassically damped systems in which the normal modes from the undamped models cannot be used to uncouple the dynamic equations, some dynamic schemes that consider the effect of damping on the transformation matrices have been defined in the state space. Rivera et al. [11] provided a dynamic condensation approach as an extension of the undamped systems of Suarez and Singh [12]. Qu et al. [13,14] developed different reduction methods according to different expression forms of the system matrices.

Although the techniques mentioned above can accomplish the FE model reduction, it requires a large amount of computational resources to construct the transformation matrix of large-scale FE models. This problem is obvious when the FE models have a large number of DOF (over several thousands). One of the ways to overcome this problem is to apply a substructuring scheme with the techniques developed. Based on the substructure algorithm, several reduction methods have been investigated. Kim and Cho [15] developed three types of reduction schemes at the subdomain level without considering the reduction problems of nonclassically damped systems. Choi et al. [16] introduced an iterative dynamic condensation by combining the iterated improved reduced system (IIRS) method with a substructuring scheme. The drawback of this method is that it also requires more computer storage to construct the transformation matrices of large-scale systems.

The objective of this study is to develop a more efficient dynamic condensation method for both undamped and damped systems with limited computer storage. Because of the convenience of the original iterative order-reduction (IOR) method [17] which only uses iterative forms of mass and transformation matrices, the required computational resources can be reduced by combining the IOR method with the substructuring scheme [16]. By extending the IOR method from displacement space to state space with complex eigenvalues, an improved IOR approach can be formed for nonclassically damped systems. Then an IOR-substructuring technique can be obtained by combining the outlined reduction procedures with the substructuring scheme. This method has two advantages. First, the improved IOR method based on the state-space model can be applied to damped structures, especially for nonclassically damped systems. Second, the IOR-substructuring method can reduce the size of the transformation matrices dramatically so that the model reduction can be accomplished with low-dimensional operation matrices and limited computer storage.

The remainder of this paper is organized as follows. Section II reviews the IOR method briefly and then describes an improved IOR method for nonclassically damped systems. Section III introduces the derivation process of the IOR-substructuring method for both undamped and damped systems. In Sec. IV, the iteration strategy of the proposed method is given, and Sec. V gives a brief discussion on the convergence and the computational efficiency of the proposed method. Two numerical examples are calculated to demonstrate the accuracy and efficiency of the proposed reduction algorithm in Sec. VI, and Sec. VII concludes the paper.

II. Original and Improved IOR Method for FE Models

As stated in [17], the original IOR technique, which uses iterative forms of mass and transformation matrices, can save computational cost to achieve accurate results for the natural frequencies of undamped structures. In this section, this method will be reviewed briefly, and then an improved IOR method based on the state-space model is proposed for nonclassically damped systems.

A. Original IOR Method for Undamped Systems

The dynamic equilibrium equations of an n-DOF system can be described in a matrix form as

$$M\ddot{x} + C\dot{x} + Kx = Df \tag{1}$$

The corresponding eigenvalue problem of the undamped structure can be expressed as

$$K\Phi = M\Phi\Lambda \tag{2}$$

where Φ denotes the eigenvectors matrix, and Λ denotes the diagonal matrix in which the eigenvalues are arranged in an ascending order on the diagonal.

Partitioning the DOF in Eq. (2) into the master DOF (retained) and slave DOF (removed), the matrices and vectors are split into submatrices and subvectors as

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{ms}^T & K_{ss} \end{bmatrix} \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{Bmatrix} = \begin{bmatrix} M_{mm} & M_{ms} \\ M_{ms}^T & M_{ss} \end{bmatrix} \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{Bmatrix} \Lambda_{mm} \tag{3}$$

where the number of master and slave DOF are n_m and $n_s (n_m + n_s = n)$, respectively. From the second set of Eq. (3), the following relation can be obtained as

$$K_{ms}^T \Phi_{mm} + K_{ss}\Phi_{sm} = (M_{ms}^T \Phi_{mm} + M_{ss}\Phi_{sm})\Lambda_{mm} \tag{4}$$

Assume that the transformation matrix of the master and slave DOF is defined as

$$\Phi_{sm} = t\Phi_{mm} \tag{5}$$

where t is a linear, nonsingular, and constant matrix.

From Eqs. (4) and (5), the expression form of t can be obtained as

$$t = -K_{ss}^{-1}K_{ms}^T + K_{ss}^{-1}(M_{ms}^T + M_{ss}t)\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} = t_G + t_d \tag{6}$$

where t_G is a constant matrix with the same expression of the Guyan method [4], and t_d is the transformation matrix in terms of t.

The relation between the master DOF and the entire set of DOF is

$$\begin{aligned} \Phi = \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{Bmatrix} = \begin{bmatrix} I \\ t \end{bmatrix} \Phi_{mm} = \begin{bmatrix} I \\ t_G + t_d \end{bmatrix} \Phi_{mm} \\ = \left(T_G + \begin{bmatrix} 0 \\ t_d \end{bmatrix} \right) \Phi_{mm} = T\Phi_{mm} \end{aligned} \tag{7}$$

where I is a unit matrix of order $n_m \times n_m$, and T_G is a combined matrix of order $n \times n_m$.

Substituting Eq. (7) into Eq. (3) and premultiplying T^T, a reduced eigenvalue problem can be obtained as

$$K_R\Phi_{mm} = M_R\Phi_{mm}\Lambda_{mm} \tag{8}$$

where

$$K_R = T^T KT = K_G + t_d^T(M_{ms}^T + M_{ss}t_G + M_{ss}t_d)\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} \tag{9a}$$

$$\begin{aligned} M_R = T^T MT = M_G + t_d^T(M_{ms}^T + M_{ss}t_G + M_{ss}t_d) \\ + (M_{ms} + t_G^T M_{ss})t_d \end{aligned} \tag{9b}$$

The form of Eq. (8) can be rearranged as

$$\begin{aligned}0 &= K_R\Phi_{mm} - M_R\Phi_{mm}\Lambda_{mm}\\ &= (K_G + t_d^T(M_{ms}^T + M_{ss}t_G + M_{ss}t_d)\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1})\Phi_{mm}\\ &\quad - (M_G + t_d^T(M_{ms}^T + M_{ss}t_G + M_{ss}t_d) + (M_{ms} + t_G^T M_{ss})t_d)\Phi_{mm}\Lambda_{mm}\\ &= K_G\Phi_{mm} - M_d\Phi_{mm}\Lambda_{mm}\end{aligned} \tag{10}$$

where

$$M_d = M_G + (M_{ms} + t_G^T M_{ss})t_d = T_G^T M T \tag{11}$$

The generalized eigenvalue problem of the reduced system is rewritten as

$$K_G\Phi_{mm} = M_d\Phi_{mm}\Lambda_{mm} \tag{12}$$

From Eq. (12), an approximation can be obtained as

$$\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} = M_d^{-1}K_G \tag{13}$$

Substituting Eq. (13) into Eq. (6), the form of the transformation matrix t can be found as

$$t = t_G + t_d = -K_{ss}^{-1}K_{ms}^T + K_{ss}^{-1}(M_{ms}^T + M_{ss}t)M_d^{-1}K_G \tag{14}$$

Because of the nonlinearity of the governing matrix above, an iterative scheme is given by

$$M_d^{[k]} = M_G + (M_{ms} + t_G^T M_{ss})t_d^{[k]} \tag{15a}$$

$$t_d^{[k+1]} = K_{ss}^{-1}(M_{ms}^T + M_{ss}t_G + M_{ss}t_d^{[k]})(M_d^{[k]})^{-1}K_G \tag{15b}$$

$$K_G\Phi_{mm}^{[k]} = M_d^{[k]}\Phi_{mm}^{[k]}\Lambda_{mm}^{[k]} \tag{15c}$$

where the original values begin with the results of the Guyan method.

Consequently, the last values of Eqs. (15a) and (15b) with sufficient iterations are adopted as the transformation and reduced mass matrices. Because of the equation $T_G^T K T_d = 0$, the reduced stiffness matrix has the following form as

$$K_G = T_G^T K T_G + T_G^T K T_d = T_G^T K T \tag{16}$$

The direction matrix of excitation force f can be obtained as

$$D_R = T_G^T D \tag{17}$$

B. Improved IOR Method for Damped Systems

According to the types of damping, damped structures can be classified as classically and nonclassically damped systems. For classically damped models, the IOR method is also valid because the damping can be assumed to be a linear combination of mass and stiffness matrices. However, there are many situations where the classically damped assumptions are invalid for nonclassically damped models. Examples of such cases are structures composed of materials with different damping characteristics in different parts, structures equipped with passive and active control systems and structures with layers of damping materials [11,13,18]. To solve the reduction problems with the effect of nonclassical damping, the state vectors, which are a combination of the velocity and displacement vectors, are often used to convert second-order differential equations into first-order equations. In this section, an improved IOR method based on the state-space model is proposed following this conversion, so that the reduced model can be guaranteed to converge to the exact values. The complex eigenvalues, eigenvectors, equivalent modal frequencies and damping ratios will be calculated, respectively. Therefore, Eq. (1) can be rewritten as

$$Gz - H\dot{z} = \bar{D}f \tag{18}$$

where the vectors z, $\dot{z}$, and system matrices G, H,and $\bar{D}$ are defined as

$$z = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \quad \dot{z} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} \quad G = \begin{bmatrix} 0 & K \\ K & C \end{bmatrix}$$
$$H = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \quad \bar{D} = \begin{bmatrix} 0 \\ D \end{bmatrix} \tag{19}$$

The corresponding eigenvalue problem of Eq. (18) can be expressed as

$$GU = HU\Omega \tag{20}$$

where the complex eigenvectors matrix U and the eigenvalues matrix Ω has the form of

$$U = \begin{bmatrix} \Psi & \Psi^* \\ \Psi\tilde{\Omega} & \Psi^*\tilde{\Omega}^* \end{bmatrix} \quad \Omega = \begin{bmatrix} \tilde{\Omega} & 0 \\ 0 & \tilde{\Omega}^* \end{bmatrix} \tag{21}$$

Based on the division of the master and slave DOF, Eq. (20) can be expressed in a partitioned form as

$$\begin{bmatrix} G_{mm} & G_{ms} \\ G_{ms}^T & G_{ss} \end{bmatrix}\begin{Bmatrix} U_{mm} \\ U_{sm} \end{Bmatrix} = \begin{bmatrix} H_{mm} & H_{ms} \\ H_{ms}^T & H_{ss} \end{bmatrix}\begin{Bmatrix} U_{mm} \\ U_{sm} \end{Bmatrix}\Omega_{mm} \tag{22}$$

where the system submatrices are defined as

$$G_{mm} = \begin{bmatrix} 0 & K_{mm} \\ K_{mm} & C_{mm} \end{bmatrix} \quad G_{ms} = G_{sm}^T = \begin{bmatrix} 0 & K_{ms} \\ K_{ms} & C_{ms} \end{bmatrix}$$
$$G_{ss} = \begin{bmatrix} 0 & K_{ss} \\ K_{ss} & C_{ss} \end{bmatrix} \tag{23a}$$

$$H_{mm} = \begin{bmatrix} K_{mm} & 0 \\ 0 & -M_{mm} \end{bmatrix} \quad H_{ms} = H_{sm}^T = \begin{bmatrix} K_{ms} & 0 \\ 0 & -M_{ms} \end{bmatrix}$$
$$H_{ss} = \begin{bmatrix} K_{ss} & 0 \\ 0 & -M_{ss} \end{bmatrix} \tag{23b}$$

$$U_{mm} = \begin{bmatrix} \Psi_{mm} & \Psi_{mm}^* \\ \Psi_{mm}\tilde{\Omega}_{mm} & \Psi_{mm}^*\tilde{\Omega}_{mm}^* \end{bmatrix}$$
$$U_{sm} = \begin{bmatrix} \Psi_{sm} & \Psi_{sm}^* \\ \Psi_{sm}\tilde{\Omega}_{sm} & \Psi_{sm}^*\tilde{\Omega}_{sm}^* \end{bmatrix} \quad \Omega = \begin{bmatrix} \tilde{\Omega}_{mm} & 0 \\ 0 & \tilde{\Omega}_{mm}^* \end{bmatrix} \tag{23c}$$

where the eigenvalues in matrix Ω are arranged in an ascending order.

The main derivation process of the improved IOR method for nonclassically damped systems is the same as that of undamped systems except that all system matrices are defined in the state space. Thus, the reduction problems for nonclassically damped FE models, such as the examples in which the linear description of damping is not suitable [11,13,18], can be solved by taking into account the effect of damping on the transformation matrix. After taking the same procedure from Eqs. (4–14), the dynamic transformation matrix is obtained by

$$t = t_G + t_d = -G_{ss}^{-1}G_{ms}^T + G_{ss}^{-1}(H_{ms}^T + H_{ss}t)H_d^{-1}G_G \tag{24}$$

where the Guyan constant matrices are defined as

$$G_G = [I \quad t_G^T]G[I \quad t_G^T]^T \tag{25a}$$

$$H_G = [I \quad t_G^T]H[I \quad t_G^T]^T \tag{25b}$$

The iterative scheme for the model reduction can be constructed as

$$H_d^{[k]} = H_G + (H_{ms} + t_G^T H_{ss})t_d^{[k]} \tag{26a}$$

$$t_d^{[k+1]} = G_{ss}^{-1}(H_{ms}^T + H_{ss}t_G + H_{ss}t_d^{[k]})(H_d^{[k]})^{-1}G_G \tag{26b}$$

$$G_G U_{mm}^{[k]} = H_d^{[k]} U_{mm}^{[k]} \Omega_{mm}^{[k]} \tag{26c}$$

where the original values begin with the results of the Guyan method.

With sufficient iterations ($k = 1, 2, \ldots, p$), the reduced system matrices in the state space are obtained as

$$G_R = G_G = \begin{bmatrix} 0 & K_R \\ K_R & C_R \end{bmatrix} \qquad H_R = H_d^{[p]} = \begin{bmatrix} K_R & 0 \\ 0 & -M_R \end{bmatrix}$$
$$\tilde{D}_R = T_G^T \bar{D} = \begin{bmatrix} 0 \\ D_R \end{bmatrix} \tag{27}$$

III. IOR-Substructuring Method for Large-Scale FE Models

The goal of the substructuring scheme is to disassemble the whole system into several substructures so that the eigenvalue problem can be solved in several single domains. Using the technique for the construction of the transformation matrix outlined above, an efficient method can be developed for the model- reduction of large-scale FE systems with limited computer storage. The FE model consists of three kinds of nodes: master, slave, and interface nodes. In this section, an IOR-substructuring method for large-scale FE systems is proposed by taking the interface nodes to be the master nodes in addition to those already selected as master nodes.

A. IOR-Substructuring Method for Undamped and Classically Damped Systems

The following formulation is the basic derivative process for large-scale undamped system reduction. It is also valid for classically damped systems because the damping does not affect the eigenvectors of models [13]. To employ the substructuring scheme, the whole structure is first divided into N substructures. Through the construction of the transformation matrix in each substructure, the eigenvalue problems can be reduced by the unification of substructure problems. Hence, the generalized eigenvalue problem of Eq. (2) can be expressed in the following partitioned form as

$$\begin{bmatrix} K_{mm} & K_{ms}^{(1)} & K_{ms}^{(2)} & \cdots & K_{ms}^{(N)} \\ K_{sm}^{(1)} & K_{ss}^{(1)} & 0 & \cdots & 0 \\ K_{sm}^{(2)} & 0 & K_{ss}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{sm}^{(N)} & 0 & 0 & \cdots & K_{ss}^{(N)} \end{bmatrix} \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm}^{(1)} \\ \Phi_{sm}^{(2)} \\ \vdots \\ \Phi_{sm}^{(N)} \end{Bmatrix} = \begin{bmatrix} M_{mm} & M_{ms}^{(1)} & M_{ms}^{(2)} & \cdots & M_{ms}^{(N)} \\ M_{sm}^{(1)} & M_{ss}^{(1)} & 0 & \cdots & 0 \\ M_{sm}^{(2)} & 0 & M_{ss}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{sm}^{(N)} & 0 & 0 & \cdots & M_{ss}^{(N)} \end{bmatrix} \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm}^{(1)} \\ \Phi_{sm}^{(2)} \\ \vdots \\ \Phi_{sm}^{(N)} \end{Bmatrix} \Lambda_{mm} \tag{28}$$

where

$$K_{mm} = \sum_{q=1}^{N} K_{mm}^{(q)} \qquad M_{mm} = \sum_{q=1}^{N} M_{mm}^{(q)} \tag{29a}$$

$$K_{sm}^{(j)} = (K_{ms}^{(j)})^T \qquad M_{sm}^{(j)} = (M_{ms}^{(j)})^T \qquad (j = 1, 2, \ldots, N) \tag{29b}$$

The eigenvalue problem of the jth ($j = 1, 2, \ldots, N$) substructure can be described as

$$\begin{bmatrix} K_{mm}^{(j)} & K_{ms}^{(j)} \\ K_{sm}^{(j)} & K_{ss}^{(j)} \end{bmatrix} \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm}^{(j)} \end{Bmatrix} = \begin{bmatrix} M_{mm}^{(j)} & M_{ms}^{(j)} \\ M_{sm}^{(j)} & M_{ss}^{(j)} \end{bmatrix} \begin{Bmatrix} \Phi_{mm} \\ \Phi_{sm}^{(j)} \end{Bmatrix} \Lambda_{mm} \tag{30}$$

The expression of the slave DOF of the jth ($j = 1, 2, \ldots, N$) substructure can be obtained from the second set of Eq. (30) as

$$\Phi_{sm}^{(j)} = -(K_{ss}^{(j)})^{-1} K_{sm}^{(j)} \Phi_{mm} + (K_{ss}^{(j)})^{-1}(M_{sm}^{(j)} \Phi_{mm} + M_{ss}^{(j)} \Phi_{sm}^{(j)}) \Lambda_{mm} \tag{31}$$

Assume that the relations between the master and slave DOF in the jth ($j = 1, 2, \ldots, N$) substructure are

$$\Phi_{sm}^{(j)} = t_j \Phi_{mm} \qquad (j = 1, 2, \ldots, N) \tag{32}$$

where t_j is a transformation matrix of order $s^{(j)} \times m$.

From Eqs. (31) and (32), the expression of the jth ($j = 1, 2, \ldots, N$) transformation matrix can be found by

$$\begin{aligned} t_j &= -(K_{ss}^{(j)})^{-1} K_{sm}^{(j)} + (K_{ss}^{(j)})^{-1}(M_{sm}^{(j)} + M_{ss}^{(j)} t_j) \Phi_{mm} \Lambda_{mm} \Phi_{mm}^{-1} \\ &= t_G^{(j)} + t_d^{(j)} \end{aligned} \tag{33}$$

The entire DOF can be described with only the master DOF as

$$\begin{aligned} &[\Phi_{mm}^T \quad (\Phi_{sm}^{(1)})^T \quad (\Phi_{sm}^{(2)})^T \quad \cdots \quad (\Phi_{sm}^{(N)})^T]^T \\ &= [I \quad t_1^T \quad t_2^T \quad \cdots \quad t_N^T]^T \Phi_{mm} = T \Phi_{mm} \end{aligned} \tag{34}$$

where I is a unit matrix of order $n_m \times n_m$, and the combined transformation matrix T is defined as

$$\begin{aligned} T = T_G + T_d = [I \quad (t_G^{(1)})^T \quad (t_G^{(2)})^T \quad \cdots \quad (t_G^{(N)})^T]^T \\ + [0 \quad (t_d^{(1)})^T \quad (t_d^{(2)})^T \quad \cdots \quad (t_d^{(N)})^T]^T \end{aligned} \tag{35}$$

Substituting Eq. (35) into Eq. (28) and premultiplying T^T, a reduced eigenvalue problem is obtained as

$$K_R \Phi_{mm} = M_R \Phi_{mm} \Lambda_{mm} \tag{36}$$

where

$$K_R = T^T K T = K_G + T_G^T K T_d + T_d^T K T_G + T_d^T K T_d \tag{37a}$$

$$M_R = T^T M T = M_G + T_G^T M T_d + T_d^T M T \tag{37b}$$

Noting in the above equations

$$T_G^T K T_d = T_d^T K T_G = 0 \tag{38a}$$

$$\begin{aligned} T_d^T K T_d = \sum_{q=1}^{N} (t_d^{(q)})^T K_{ss}^{(q)} t_d^{(q)} = \sum_{q=1}^{N} (t_d^{(q)})^T (M_{sm}^{(q)} \\ + M_{ss}^{(q)} t_q) \Phi_{mm} \Lambda_{mm} \Phi_{mm}^{-1} \end{aligned} \tag{38b}$$

$$T_G^T M T_d = \sum_{q=1}^{N} (M_{ms}^{(q)} + (t_G^{(q)})^T M_{ss}^{(q)}) t_d^{(q)} \tag{38c}$$

$$T_d^T MT = \sum_{q=1}^{N} (t_d^{(q)})^T (M_{sm}^{(q)} + M_{ss}^{(q)} t_q) \tag{38d}$$

Equation (36) can be arranged as

$$\begin{aligned} 0 &= K_R\Phi_{mm} - M_R\Phi_{mm}\Lambda_{mm} \\ &= \left(K_G + \sum_{q=1}^{N}(t_d^{(q)})^T(M_{sm}^{(q)} + M_{ss}^{(q)}t_q)\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1}\right)\Phi_{mm} \\ &\quad - \left(M_G + \sum_{q=1}^{N}(M_{ms}^{(q)} + (t_G^{(q)})^T M_{ss}^{(q)})t_d^{(q)} \right. \\ &\quad \left. + \sum_{q=1}^{N}(t_d^{(q)})^T(M_{sm}^{(q)} + M_{ss}^{(q)}t_q)\right)\Phi_{mm}\Lambda_{mm} \\ &= K_G\Phi_{mm} - M_d\Phi_{mm}\Lambda_{mm} \end{aligned} \tag{39}$$

where

$$M_d = M_G + \sum_{q=1}^{N}(M_{ms}^{(q)} + (t_G^{(q)})^T M_{ss}^{(q)})t_d^{(q)} = T_G^T MT \tag{40}$$

The generalized eigenvalue problem of the reduced system changes into

$$K_G\Phi_{mm} = M_d\Phi_{mm}\Lambda_{mm} \tag{41}$$

The approximation of the above equation is found by

$$\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} = M_d^{-1}K_G \tag{42}$$

Substituting Eq. (42) into Eq. (33), the transformation matrix of the jth $(j = 1, 2, \ldots, N)$ substructure can be obtained as

$$t_j = -(K_{ss}^{(j)})^{-1}K_{sm}^{(j)} + (K_{ss}^{(j)})^{-1}(M_{sm}^{(j)} + M_{ss}^{(j)}t_j)M_d^{-1}K_G = t_G^{(j)} + t_d^{(j)} \tag{43}$$

Like Eq. (15), an iterative scheme is given by $(j = 1, 2, \ldots, N)$

$$M_d^{[k]} = M_G + \sum_{q=1}^{N}(M_{ms}^{(q)} + (t_G^{(q)})^T M_{ss}^{(q)})(t_d^{(q)})^{[k]} \tag{44a}$$

$$(t_d^{(j)})^{[k+1]} = (K_{ss}^{(j)})^{-1}(M_{sm}^{(j)} + M_{ss}^{(j)}t_G^{(j)} + M_{ss}^{(j)}(t_d^{(j)})^{[k]})(M_d^{[k]})^{-1}K_G \tag{44b}$$

$$K_G\Phi_{mm}^{[k]} = M_d^{[k]}\Phi_{mm}^{[k]}\Lambda_{mm}^{[k]} \tag{44c}$$

where the original values begin with the results of the Guyan method. With sufficient iterations $(k = 1, 2, \ldots, p)$, the reduced system matrices are obtained by

$$M_R = M_d^{[p]} \qquad K_R = K_G \qquad D_R = T_G^T D \tag{45}$$

B. IOR-Substructuring Method for Nonclassically Damped Systems

The above technique for undamped and classically damped systems can also be extended to nonclassically damped systems. As mentioned in Sec. II, the system matrices will be doubled in order, and the process will refer to the complex operation in the state space. The eigenvalue problem can be expressed as

$$\begin{bmatrix} G_{mm} & G_{ms}^{(1)} & G_{ms}^{(2)} & \cdots & G_{ms}^{(N)} \\ G_{sm}^{(1)} & G_{ss}^{(1)} & 0 & \cdots & 0 \\ G_{sm}^{(2)} & 0 & G_{ss}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{sm}^{(N)} & 0 & 0 & \cdots & G_{ss}^{(N)} \end{bmatrix} \begin{Bmatrix} U_{mm} \\ U_{sm}^{(1)} \\ U_{sm}^{(2)} \\ \vdots \\ U_{sm}^{(N)} \end{Bmatrix} = \begin{bmatrix} H_{mm} & H_{ms}^{(1)} & H_{ms}^{(2)} & \cdots & H_{ms}^{(N)} \\ H_{sm}^{(1)} & H_{ss}^{(1)} & 0 & \cdots & 0 \\ H_{sm}^{(2)} & 0 & H_{ss}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{sm}^{(N)} & 0 & 0 & \cdots & H_{ss}^{(N)} \end{bmatrix} \begin{Bmatrix} U_{mm} \\ U_{sm}^{(1)} \\ U_{sm}^{(2)} \\ \vdots \\ U_{sm}^{(N)} \end{Bmatrix} \Omega_{mm} \tag{46}$$

where

$$G_{mm} = \sum_{q=1}^{N} G_{mm}^{(q)} \qquad H_{mm} = \sum_{q=1}^{N} H_{mm}^{(q)} \tag{47a}$$

$$G_{sm}^{(j)} = (G_{ms}^{(j)})^T \qquad H_{sm}^{(j)} = (H_{ms}^{(j)})^T \qquad (j = 1, 2, \ldots, N) \tag{47b}$$

The eigenvalue problem of the jth $(j = 1, 2, \ldots, N)$ substructure can be described as

$$\begin{bmatrix} G_{mm}^{(j)} & G_{ms}^{(j)} \\ G_{sm}^{(j)} & G_{ss}^{(j)} \end{bmatrix} \begin{Bmatrix} U_{mm} \\ U_{sm}^{(j)} \end{Bmatrix} = \begin{bmatrix} H_{mm}^{(j)} & H_{ms}^{(j)} \\ H_{sm}^{(j)} & H_{ss}^{(j)} \end{bmatrix} \begin{Bmatrix} U_{mm} \\ U_{sm}^{(j)} \end{Bmatrix} \Omega_{mm} \tag{48}$$

With the same derivation process of Eqs. (31–42), the transformation matrix of the jth $(j = 1, 2, \ldots, N)$ substructure can be expressed as

$$t_j = -(G_{ss}^{(j)})^{-1}G_{sm}^{(j)} + (G_{ss}^{(j)})^{-1}(H_{sm}^{(j)} + H_{ss}^{(j)}t_j)H_d^{-1}G_G = t_G^{(j)} + t_d^{(j)} \tag{49}$$

Like Eq. (44), the iterative condensation scheme for the reduced system is given by $(j = 1, 2, \ldots, N)$

$$H_d^{[k]} = H_G + \sum_{q=1}^{N}(H_{ms}^{(q)} + (t_G^{(q)})^T H_{ss}^{(q)})(t_d^{(q)})^{[k]} \tag{50a}$$

$$(t_d^{(j)})^{[k+1]} = (G_{ss}^{(j)})^{-1}(H_{sm}^{(j)} + H_{ss}^{(j)}t_G^{(j)} + H_{ss}^{(j)}(t_d^{(j)})^{[k]})(H_d^{[k]})^{-1}G_G \tag{50b}$$

$$G_G\Phi_{mm}^{[k]} = H_d^{[k]}\Phi_{mm}^{[k]}\Lambda_{mm}^{[k]} \tag{50c}$$

With sufficient iterations $(k = 1, 2, \ldots, p)$, the reduced system matrices in the state space are obtained by

$$G_R = G_G = \begin{bmatrix} 0 & K_R \\ K_R & C_R \end{bmatrix} \qquad H_R = H_d^{[p]} = \begin{bmatrix} K_R & 0 \\ 0 & -M_R \end{bmatrix}$$
$$\tilde{D}_R = T_G^T\bar{D} = \begin{bmatrix} 0 \\ D_R \end{bmatrix} \tag{51}$$

IV. Iteration Strategy of the IOR-Substructuring Reduction Method

In Secs. II and III, the reduced system matrices of undamped and classically damped systems were obtained by the same condensation procedure described in the displacement space, and the reduced system matrices of nonclassically damped systems could be found with the proposed method based on the state-space model. The main iterative strategies for the model reduction of undamped and damped systems are listed as follows:

1) Divide the whole FE model into N substructures.

2) Select the master and slave DOF, and rewrite the system matrices to formulate the partitioned matrices ($j = 1, 2, \ldots, N$).

3) Construct the reduced matrices of the substructures by the Guyan method.

For case I, undamped and classically damped systems ($j = 1, 2, \ldots, N$),

$$t_G^{(j)} = -(K_{ss}^{(j)})^{-1} K_{sm}^{(j)}$$
$$T_G^T = [I \quad (t_G^{(1)})^T \quad (t_G^{(2)})^T \quad \cdots \quad (t_G^{(N)})^T]$$
$$M_G = T_G^T M T_G \qquad K_G = T_G^T K T_G \tag{52a}$$

For case II, nonclassically damped systems ($j = 1, 2, \ldots, N$),

$$t_G^{(j)} = -(G_{ss}^{(j)})^{-1} G_{sm}^{(j)} \qquad T_G^T = [I \quad (t_G^{(1)})^T \quad (t_G^{(2)})^T \quad \cdots \quad (t_G^{(N)})^T] \tag{52b}$$

4) Iterations are listed as follows ($k = 1, 2, \ldots, p$).

a) Assume that the initial values of the jth ($j = 1, 2, \ldots, N$) transformation matrix are

$$(t_d^{(j)})^{[0]} = 0 \tag{53}$$

b) Calculate the reduced system matrix as follows.

For case I,

$$M_d^{[k-1]} = M_G + \sum_{q=1}^{N} (M_{ms}^{(q)} + (t_G^{(q)})^T M_{ss}^{(q)})(t_d^{(q)})^{[k-1]} \tag{54a}$$

For case II,

$$H_d^{[k-1]} = H_G + \sum_{q=1}^{N} (H_{ms}^{(q)} + (t_G^{(q)})^T H_{ss}^{(q)})(t_d^{(q)})^{[k-1]} \tag{54b}$$

c) Construct the next values of the transformation matrices of the jth ($j = 1, 2, \ldots, N$) substructure as follows.

For case I,

$$t_j^{[k-1]} = t_G^{(j)} + (t_d^{(j)})^{[k-1]}$$
$$(t_d^{(j)})^{[k]} = (K_{ss}^{(j)})^{-1} (M_{sm}^{(j)} + M_{ss}^{(j)} t_j^{[k-1]})(M_d^{[k-1]})^{-1} K_G \tag{55a}$$

For case II,

$$t_j^{[k-1]} = t_G^{(j)} + (t_d^{(j)})^{[k-1]}$$
$$(t_d^{(j)})^{[k]} = (G_{ss}^{(j)})^{-1} (H_{sm}^{(j)} + H_{ss}^{(j)} t_j^{[k-1]})(H_d^{[k-1]})^{-1} G_G \tag{55b}$$

d) Calculate the eigenvalues of the reduced system as follows.

For case I,

$$K_G \Phi_{mm}^{[k-1]} = M_d^{[k-1]} \Phi_{mm}^{[k-1]} \Lambda_{mm}^{[k-1]} \tag{56a}$$

For case II,

$$G_G U_{mm}^{[k-1]} = H_d^{[k-1]} U_{mm}^{[k-1]} \Omega_{mm}^{[k-1]} \tag{56b}$$

e) Check the convergence of the iterations using the following criterion.

For case I,

$$|\omega_i^{[k]} - \omega_i^{[k-1]}| / \omega_i^{[k]} \le \varepsilon_1 \tag{57a}$$

For case II,

$$|\Gamma(\Omega_i^{[k]}) - \Gamma(\Omega_i^{[k-1]})| / |\Gamma(\Omega_i^{[k]})| \le \varepsilon_2, \qquad \Gamma = (R_e, I_m) \tag{57b}$$

where $\omega_i^{[k]}$ represents the ith natural frequencies with the kth iteration number in case I; $R_e(\Omega_i^{[k]})$ and $I_m(\Omega_i^{[k]})$ denote the real and imaginary parts of the ith damped frequencies in case II, respectively; and ε_1 and ε_2 are the given error tolerances. If the given conditions are satisfied, exit the iterative loop.

5) Obtain the reduced system matrices as follows.

For case I,

$$M_R = M_d^{[p]} \qquad K_R = K_G \qquad D_R = T_G^T D \tag{58a}$$

For case II,

$$H_R = H_d^{[p]} = \begin{bmatrix} K_R & 0 \\ 0 & -M_R \end{bmatrix} \qquad G_R = G_G = \begin{bmatrix} 0 & K_R \\ K_R & C_R \end{bmatrix}$$
$$\tilde{D}_R = T_G^T \bar{D} = \begin{bmatrix} 0 \\ D_R \end{bmatrix} \tag{58b}$$

V. Discussion on the Convergence and Computational Efficiency of the IOR-Substructuring Method

It is essential to consider the convergence of the proposed IOR-substructuring method. The main reduction procedures in the IOR-substructuring method in Sec. III are the same as that in the IOR method, which imply that if the master DOF are selected identically in the construction of the transformation matrices, the reduced FE model using the IOR-substructuring method has the same eigenvalue problems as the model using the IOR method. Because the transformation matrix is the relationship between the master and slave DOF, the characteristics of the slave DOF in each substructure can be transferred to the master DOF. Therefore, the convergence problem of the IOR-substructuring procedure is identical to that resulting from the original IOR method. Meanwhile, due to the operation of all master DOF in each substructuring scheme, the fundamental characteristics of structures are emphatically retained, implying that the iteration of the IOR-substructuring method will converge to the more accurate values. The proposed substructuring reduction scheme is also applicable to the nonclassically damped system in the same manner [16].

Computational efficiency is another consideration for model reduction of large-scale systems. The IOR method has been proved to be with less computational resources and fast computational speed in [17]. However, compared with the original IOR technique, the IOR-substructuring approach for undamped FE models is more efficient by decreasing the corresponding multiplications involved in the implementation through avoiding the irrelevant relations of the slave DOF in each substructure. Being similar to the multiplication count in the original IOR method and IOR-substructuring method for undamped FE models, the improved IOR approach and IOR-substructuring approach for nonclassically damped systems also have high computational efficiency, except for the double size of all operation matrices in the state space.

VI. Numerical Examples

The first example is a practical shell structure that is subjected to pressure and is fixed in the ground, as shown in Fig. 1. It is adopted to illustrate the improvement of computation efficiency by the division of undamped subsystems. From the bottom to the top, Young's moduli of the different materials are 0.7 GPa and 2.06 GPa, the mass densities are 2.700×10^3 and 7.850×10^3 kg/m^3, and the Poisson ratio is 0.3. All of the thicknesses are 0.3 m. As shown, the structure is discretized into 6053 nodes, with 36,318 DOF in total. Each node has six DOF in the x, y, and z directions and rotation around the x, y, and z axes.

The reduced results may highly depend on the selection of master DOF. Some selection strategies have been studied, such as the traditional sequential elimination method [19], the analytical selection scheme [20], the slave DOF compensation method [21], and the energy estimation method [22]. In this study, the node-based arbitrary selection method proposed by Choi et al. [16] is adopted.

To apply the proposed substructuring reduction technique to the FE model, the shell structure is disassembled into five substructures along the z axis at −1.3660, −0.5000, 0.5000, and 1.4682 m, as

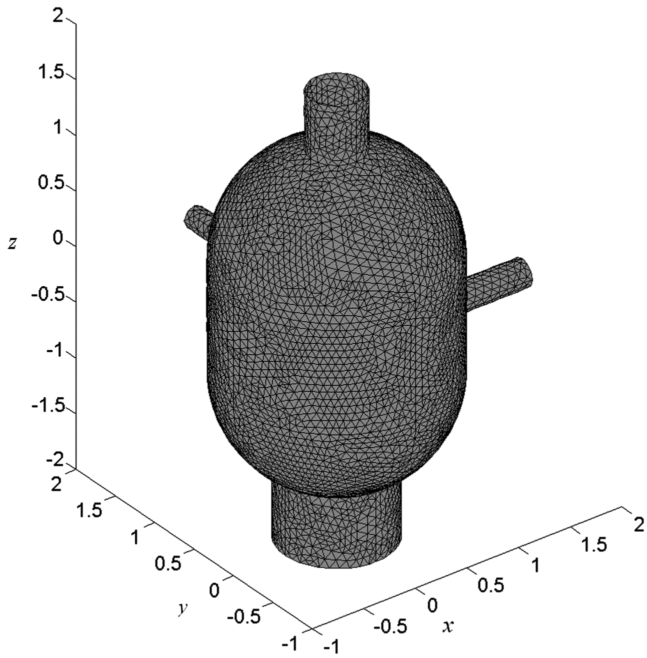

Fig. 1 Shell structure.

shown in Fig. 1. For convenience, the master nodes only include the interface nodes. The nodes of the final reduced system are only 3.63% of the original system. Table 1 shows the number of nodes and the size of the transformation matrices in the whole system and each substructure. Table 2 shows the convergence of the first nine natural frequencies with the IOR-substructuring method.

For comparison, the former 15 natural frequencies of the reduced FE model will be calculated with the IOR method [17], the IIRS-substructuring method [16], and the method presented here. To compare the accuracy of the reduced models resulting from the different techniques, the absolute relative errors are calculated as

$$\text{Error} = |\omega_{\text{red}} - \omega_{\text{full}}|/\omega_{\text{full}} \times 100\% \tag{59}$$

where ω_{full} and ω_{red} are the natural frequencies calculated from the whole system and reduced FE model, respectively. Under the same conditions, the relative errors of the natural frequencies, the maximum size of the transformation matrices and the computation time are listed in Table 3 after nine iterations. The results show that the proposed reduction method requires less computation time and computer storage to obtain more accurate results for the natural frequencies, implying that the IOR-substructuring method is computationally more efficient and accurate than the other two methods.

The second example is a truss structure for a NASA space experiment [23], as shown in Fig. 2, chosen to illustrate the efficiency of the proposed method in obtaining the reduced eigenvalues of nonclassically damping systems. The material is aluminum alloy. Young's modulus is 0.727 GPa, the mass density is 3.1×10^3 kg/m^3, and the Poisson ratio is 0.3. All the cross-sectional areas of members are 7.9×10^{-3} m^2, and all the moments of inertia are 4.9087×10^{-6} m^4. The lengths of the horizontal, vertical, and diagonal beams are 5, 5, and 7.071 m, respectively. The damping characteristic is constructed as

$$C = \alpha \times \text{diag}(K_{11}, K_{22}, K_{33}, \ldots, K_{nn}) \tag{60}$$

The structural model is discretized to 372 beam elements, each of which is 2.5 m in horizontal and vertical beams and 2.357 m in diagonal beams. This results in 268 nodes and 1584 DOF in total. Each node has six DOF in the x, y, and z directions and rotation around x, y, and z axes. The whole structure is disassembled into four substructures. The interface nodes are located at 15, 30, and 45 m along the z axis. The geometric range in the z direction, number of elements, nodes, and interface nodes in the whole system and each substructure are listed in Table 4.

To investigate the influence of the number of master DOF, three cases for the selection of masters are considered. For case I, only the interface nodes are chosen as the masters, for simplicity. For case II, the other nodes located at 10, 25, 40, and 55 m along the z axis are also selected as masters. The nodes located at 5, 10, 20, 25, 35, 40, 50, 55, and 60 m along the z axis are selected as masters in case III. After 10 iterations, the results are listed in Table 5 ($\alpha = 1.0 \times 10^{-6}$). From the table, we can see that the results of the reduced frequencies are close to the exact values with the increasing number of master DOF.

For the purpose of comparison with the reduced model, the truss structure is analyzed using the IIRS-substructuring method [16] and the proposed method according to the selected master nodes in case II. Because all eigenvalues are complex conjugate pairs, only one value of each pair is considered. The eigenvalue corresponding to the jth mode is denoted as

$$\Omega_j = -\alpha_j \pm i\beta_j \tag{61}$$

Table 1 Number of nodes and size of the transformation matrix in the whole system and substructures

	Total nodes	Master nodes	Slave nodes	Transformation matrix
Full system	36,318	220	5793	$34,758 \times 1320$
Substructures:				
Sub 1	79	40	723	4338×1320
Sub 2	108	120	1409	8454×1320
Sub 3	320	160	1860	$11,160 \times 1320$
Sub 4	96	100	1563	9378×1320
Sub 5	112	20	238	1428×1320

Table 2 Convergence of the jth ($j = 1, 2, \ldots, 9$) natural frequencies

	Natural frequencies, Hz								
Iteration	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
1	33.5675	33.6092	75.0361	177.4649	200.1936	200.7461	365.1615	395.2089	594.9787
2	33.5503	33.5932	74.6819	175.0601	196.6211	197.7409	309.1159	322.6969	343.2576
3	33.5503	33.5932	74.6775	174.8826	196.2616	197.5383	290.0421	290.2271	305.4856
4	33.5503	33.5932	74.6775	174.8812	196.2577	197.5364	289.2735	289.3626	305.2507
5	33.5503	33.5932	74.6775	174.8811	196.2577	197.5364	289.2687	289.3580	305.2493
6	33.5503	33.5932	74.6775	174.8811	196.2577	197.5364	289.2687	289.3579	305.2493
Exact	33.5503	33.5932	74.6775	174.8811	196.2577	197.5364	289.2687	289.3579	305.2493

Table 3 Comparisons of the results with different reduction methods after nine iterations

Mode	Relative errors of natural frequencies		
	IOR method	IIRS-substructuring method	IOR-substructuring method
$j = 1$	1.2041×10^{-6}	1.0745×10^{-8}	2.6547×10^{-8}
$j = 2$	9.9037×10^{-7}	9.7398×10^{-8}	8.4159×10^{-8}
$j = 3$	2.9442×10^{-5}	1.0074×10^{-9}	7.4700×10^{-9}
$j = 4$	3.0195×10^{-4}	1.5157×10^{-6}	9.8922×10^{-9}
$j = 5$	1.1626×10^{-4}	3.5495×10^{-6}	2.7384×10^{-8}
$j = 6$	3.8274×10^{-5}	1.9541×10^{-6}	3.6488×10^{-8}
$j = 7$	3.0856×10^{-4}	4.8660×10^{-4}	3.8676×10^{-8}
$j = 8$	1.9086×10^{-4}	5.9097×10^{-4}	1.2748×10^{-9}
$j = 9$	3.2169×10^{-4}	1.5376×10^{-4}	4.9221×10^{-8}
$j = 10$	2.0127×10^{-4}	1.8462×10^{-4}	4.5572×10^{-8}
$j = 11$	5.1534×10^{-5}	1.3008×10^{-5}	1.3519×10^{-8}
$j = 12$	7.3626×10^{-5}	1.9033×10^{-5}	4.1535×10^{-10}
$j = 13$	3.8808×10^{-5}	7.7614×10^{-5}	3.6082×10^{-8}
$j = 14$	9.9799×10^{-5}	3.4309×10^{-5}	1.8454×10^{-10}
$j = 15$	6.7282×10^{-5}	3.1752×10^{-5}	1.0485×10^{-8}
Maximum size	34758×1320	11160×1320	11160×1320
Elapsed time(s)	5.0946×10^{3}	1.3759×10^{4}	3.6754×10^{3}

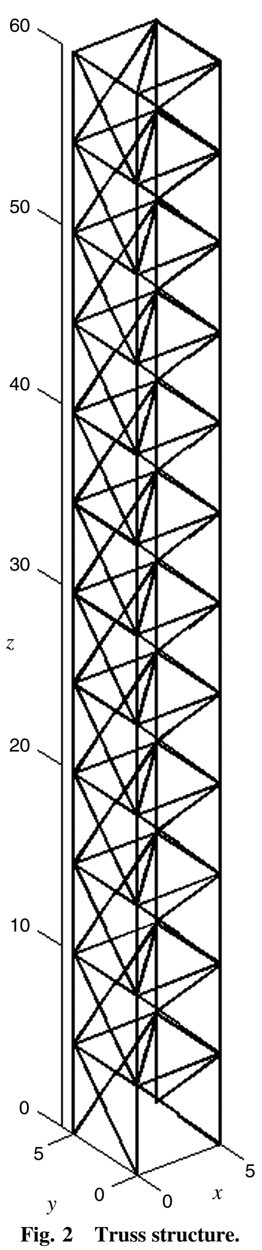

Fig. 2 Truss structure.

The equivalent modal frequencies and damping ratios can be calculated by

$$\omega_j = (\alpha_j^2 + \beta_j^2)^{1/2}/2\pi \qquad \xi_j = \alpha_j/2\pi\omega_j \qquad (j = 1, 2, \ldots, m) \tag{62}$$

The absolute relative errors of the equivalent natural frequencies and damping ratios are defined as

$$\text{Error} = |\zeta_{\text{red}} - \zeta_{\text{full}}|/\zeta_{\text{full}} \times 100\% \qquad (\zeta = \omega, \xi) \tag{63}$$

where ζ_{red} and ζ_{full} represent the modal frequencies and damping ratios of the whole and reduced system, respectively. After 10 iterations, the relative errors of the equivalent modal frequencies and the damping ratios and the elapsed time are given in Table 6. The results show that the proposed reduction method is not only more computationally efficient than the IIRS-substructuring method but can also achieve a higher accuracy with the same iterations.

To investigate the effect of substructure divisions, the truss structure is disassembled into 1 (improved IOR), 2, 3, and 4 substructures, along the z axis. For simplicity, the master nodes in case I are selected as the masters. The number of master and slave nodes with different substructures and the computation time for the calculation of the transformation matrices are listed in Table 7 after nine iteration times. It is obvious that the computation time decreases with increasing substructure divisions, while the number of slave nodes in each substructure is larger than that of master nodes. All the computation of this paper is executed by MATLAB code, but it is also easy to implement the proposed numerical methods in any other programming language, such as FORTRAN, C, etc.

VII. Conclusions

By combining the IOR scheme with the substructuring scheme, an iterative model-reduction method for both undamped and damped systems was presented. For undamped systems, a new IOR-substructuring technique was formed by adopting the substructuring scheme in the original IOR method. Then an improved IOR approach and an IOR-substructuring approach for nonclassically damped systems were obtained by extending the outlined IOR method and IOR-substructuring method from displacement space to state space with complex eigenvalues. As compared with the current order-reduction methods, the proposed technique not only requires less computational resources but also achieves more accurate results. Two numerical examples were performed to show the effectiveness of the proposed method. The corresponding eigenvalue problems of the undamped FE model were studied in the first example. The second example was presented to demonstrate the validity of the proposed method for nonclassically damped systems. The

Table 4 Information for the whole FE model and substructures

	Geometric Range (m)	Number of elements	Number of nodes	Number of DOF	Number of interface node
Full system	0–60	372	268	1608	30
Substructures					
Sub 1	0–15	93	70	420	10
Sub 2	15–30	104	76	456	20
Sub 3	30–45	104	76	456	20
Sub 4	45–60	104	76	456	10

Table 5 Eigenvalue comparison of the reduced system according to the different master DOF

	Eigenvalues			
Mode	Case I	Case II	Case III	Exact
1	$-1.1451 \pm 5.5784i$	$-1.1451 \pm 5.5784i$	$-1.1451 \pm 5.5784i$	$-1.1451 \pm 5.5784i$
2	$-1.1428 \pm 5.5853i$	$-1.1428 \pm 5.5853i$	$-1.1428 \pm 5.5853i$	$-1.1428 \pm 5.5853i$
3	$-1.4150 \pm 23.3776i$	$-1.4150 \pm 23.3776i$	$-1.4150 \pm 23.3776i$	$-1.4150 \pm 23.3776i$
4	$-1.0417 \pm 27.4002i$	$-1.0411 \pm 27.3985i$	$-1.0410 \pm 27.3984i$	$-1.0410 \pm 27.3984i$
5	$-0.9241 \pm 28.5558i$	$-0.9233 \pm 28.5541i$	$-0.9233 \pm 28.5540i$	$-0.9233 \pm 28.5540i$
6	$-0.0560 \pm 37.4882i$	$-0.0660 \pm 37.3989i$	$-0.0660 \pm 37.3959i$	$-0.0660 \pm 37.3959i$
7	$-0.1148 \pm 37.8450i$	$-0.4157 \pm 37.4363i$	$-0.4153 \pm 37.4295i$	$-0.4153 \pm 37.4295i$
8	$-0.6283 \pm 37.9915i$	$-0.1132 \pm 37.6319i$	$-0.1129 \pm 37.6278i$	$-0.1129 \pm 37.6278i$
9	$-0.9015 \pm 38.0017i$	$-0.8941 \pm 37.8193i$	$-0.8935 \pm 37.8166i$	$-0.8935 \pm 37.8166i$
10	$-0.5316 \pm 38.1043i$	$-0.5027 \pm 37.8851i$	$-0.5025 \pm 37.8822i$	$-0.5025 \pm 37.8822i$
11	$-0.3628 \pm 38.3132i$	$-0.6056 \pm 37.8908i$	$-0.6048 \pm 37.8877i$	$-0.6047 \pm 37.8877i$
12	$-0.2688 \pm 38.4120i$	$-0.1903 \pm 38.0105i$	$-0.1908 \pm 38.0061i$	$-0.1908 \pm 38.0061i$
13	$-0.7938 \pm 38.4823i$	$-0.3492 \pm 38.1002i$	$-0.3497 \pm 38.0975i$	$-0.3497 \pm 38.0975i$
14	$-0.7741 \pm 38.6778i$	$-0.7960 \pm 38.2162i$	$-0.6821 \pm 38.2115i$	$-0.6821 \pm 38.2115i$
15	$-0.9482 \pm 38.8409i$	$-0.6861 \pm 38.2175i$	$-0.7963 \pm 38.2122i$	$-0.7963 \pm 38.2122i$
16	$-0.8305 \pm 38.8612i$	$-0.3361 \pm 38.4413i$	$-0.3369 \pm 38.4369i$	$-0.3369 \pm 38.4369i$
17	$-0.9561 \pm 38.9193i$	$-0.8308 \pm 38.6106i$	$-0.8311 \pm 38.6026i$	$-0.8311 \pm 38.6026i$
18	$-0.7674 \pm 39.1145i$	$-0.7296 \pm 38.6397i$	$-0.7264 \pm 38.6337i$	$-0.7264 \pm 38.6337i$
19	$-0.7373 \pm 39.1905i$	$-0.9541 \pm 38.7129i$	$-0.9488 \pm 38.7085i$	$-0.9488 \pm 38.7085i$
20	$-0.8731 \pm 39.2911i$	$-0.9177 \pm 38.7185i$	$-0.9197 \pm 38.7179i$	$-0.9197 \pm 38.7179i$

Table 6 Comparison of the first 20 modal frequencies and damping ratios with different methods

	IIRS-substructuring method		IOR-substructuring method	
Mode	Relative error of frequencies	Relative errors of ratios	Relative error of frequencies	Relative errors of ratios
1	7.8201×10^{-7}	3.8698×10^{-7}	4.5688×10^{-9}	7.0014×10^{-8}
2	4.3426×10^{-7}	8.4257×10^{-7}	7.7949×10^{-9}	1.7749×10^{-8}
3	1.9551×10^{-4}	6.1969×10^{-4}	8.1384×10^{-8}	2.9200×10^{-6}
4	7.4242×10^{-4}	1.5709×10^{-4}	2.4998×10^{-6}	2.7577×10^{-5}
5	6.3532×10^{-4}	1.4778×10^{-3}	7.4188×10^{-7}	6.5073×10^{-6}
6	5.0945×10^{-3}	9.4103×10^{-2}	5.7904×10^{-5}	6.0332×10^{-4}
7	5.3312×10^{-3}	3.1864×10^{-2}	1.6237×10^{-4}	2.6693×10^{-4}
8	5.8490×10^{-3}	1.4554×10^{-1}	8.1670×10^{-5}	1.5964×10^{-3}
9	3.8732×10^{-3}	1.0034×10^{-2}	6.2454×10^{-5}	3.8775×10^{-6}
10	2.9568×10^{-3}	1.6385×10^{-1}	6.9107×10^{-5}	2.5112×10^{-4}
11	3.2639×10^{-3}	2.1412×10^{-1}	7.2917×10^{-5}	1.4611×10^{-3}
12	5.2002×10^{-3}	7.6024×10^{-1}	9.1356×10^{-5}	2.4562×10^{-3}
13	3.6323×10^{-3}	4.5112×10^{-1}	6.7605×10^{-5}	1.4513×10^{-3}
14	3.9183×10^{-3}	1.8403×10^{-1}	1.5430×10^{-4}	1.6702×10^{-1}
15	5.4137×10^{-3}	1.0846×10^{-1}	6.2096×10^{-5}	1.3904×10^{-1}
16	4.2040×10^{-3}	4.2866×10^{-2}	9.5433×10^{-5}	1.8801×10^{-3}
17	4.9995×10^{-3}	3.2540×10^{-2}	1.9228×10^{-4}	1.1479×10^{-3}
18	5.3099×10^{-3}	3.0514×10^{-1}	1.3966×10^{-4}	3.7982×10^{-3}
19	3.7809×10^{-3}	2.2147×10^{-1}	5.1753×10^{-5}	8.7548×10^{-4}
20	3.8383×10^{-3}	2.6334×10^{-2}	4.9010×10^{-5}	2.2066×10^{-4}
Elapsed time(s)	193.9840		120.5000	

Table 7 Comparison of computation time and the number of masters and slaves with different substructures

		IOR-substructuring method		
	Improved IOR method	Two subs	Three subs	Four subs
Number of master nodes	30	30	30	30
Number of slave nodes	234	sub1: 112, sub2: 122	sub1: 112, sub2: 56, sub3: 66	Sub1: 56, sub2: 56, sub3: 56, sub4: 66
Elapsed time(s)	326.5940	118.8280	82.1870	54.4690

conclusions can be obtained from the results of the numerical examples. First, the improved IOR scheme can be used to accomplish the model reduction of nonclassically damped systems by taking into account the damping in the transformation matrix. Second, the size of the condensation matrices of large-scale structures can be obtained with limited computer storage. Finally, the results show that the presented method is more efficient and accurate than the IIRS-substructuring method. However, more research is required to improve the effectiveness of the model in substructure divisions.

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